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On the operator product expansion in non-commutative quantum field theory

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Abstract: Motivated by the mixing of UV and IR effects, we test the OPE formula in non-commutative field theory. First we look at the renormalization of local composite operators, identifying some of their characteristic IR/UV singularities. Then we find that the product of two fields in general cannot be described by a series expansion of single local operator insertions.

Keywords: Non-Commutative Geometry, Renormalization Regularization and Renormalons.

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Contents

1. Introduction

To send the location of two operators to the same spacetime point is a singular process in local Quantum Field Theory (QFT) if the ultraviolet (UV) cut-off mass is not finite. This combination of locality with UV divergences is represented by the Operator Product Expansion (OPE) [1], where the short distance dependence between the locations of the two operators is encoded in the Wilson coefficients. The very nature of the OPE is much in the spirit of the Wilson renormalization group approach to field theories: the decoupling of scales allows us to codify the short distance effects in the coefficients multiplying a series of insertions of local composite operators. Another way to see the divergences associated with the Wilson coefficients is to observe that sending the cutoff to infinity and locating the operators at the same spacetime point are two different limits which in general do not commute.

Recently, Non-commutative Quantum Field Theory (NCQFT) has been the subject of an intense research, either by using field theory methods [2]–[24] or by explot-ing its embeding into string theory [25]–[45]. Up to now, one of its most intriguing features is the mixing of UV with IR effects [7]. Since the OPE is such an important and characteristic property of local QFT, it is very natural to test it in NCQFT. Due to the UV/IR mixing of scales, something different should happen in the process of sending operators to the same spacetime point.

The simplest non-trivial framework to analyze this is via a single scalar with a cubic interaction,

$$\mathcal{L} = \frac{1}{2}(\partial\phi)\star(\partial\phi) + \frac{1}{2}m^2\phi\star\phi + \frac{g}{6}\phi\star\phi\star\phi\,. \tag{1.1}$$

We will work in six dimensions, with $\theta^{0i} = 0$, and Wick-rotate to the euclidian signature. Our analysis will be perturbative and mainly restricted to one-loop. But we expect our results to hold at higher loops, provided that the renormalization program à la Dyson (without keeping a finite cutoff) can be extended at a multi-loop level.

In section 2 we analyze the insertion of single renormalized composite operators in non-commutative scalar field theories, focusing on their IR/UV singularities. In section 3 we look for singularities related to the product of two fundamental fields in the six-dimensional ϕ^3 field theory. We see that $\theta^{ij} \neq 0$ modifies the analytic structure of the Green functions, preventing the possibility of replacing the product of two fundamental fields by a series expansion of single local composite operator insertions. The possible stringy explanations of this work is left for the future.

2. Renormalization of composite operators in NCQFT

UV divergences. Before testing the OPE in NCQFT, we should first look at the properties of the local composite operators. By local composite operator we understand an arbitrary number of fields and/or derivatives of fields, all evaluated at the same spacetime point. In local QFT, the insertion of a bare composite operator $\mathcal{O}_0$ in a general n-point Green function carries UV divergences which are intrinsic to the composite operator and require their own renormalization. The symbol for the renormalized composite operator, i.e. the one whose insertion in a general n-point Green function is finite at infinite cutoff, is expressed by $[\mathcal{O}]$, to distinguish from $\mathcal{O}$, the simple product of renormalized fields, which is a different object.

Furthermore, the renormalization produces operator mixing,

$$[\mathcal{O}_i](\mu) = \sum_{j;\ d_j \leq d_i} Z_{ij}\left(\frac{\Lambda}{\mu}; g(\mu)\right) \Lambda^{d_i - d_j} \mathcal{O}_{0,j}(\Lambda)\,, \tag{2.1}$$

where the sum only involves operators whose canonical mass dimension d_j is smaller or equal than d_i. Λ is the UV cutoff, μ the renormalization scale and $g(\mu)$ the renormalized coupling constant.

In principle any local composite operator is a linear combination of $\{\phi^2, (\partial^2\phi)\phi, \dots\}$. But it is very likely that in NCQFT it is more convenient to use the basis of operators where any product of fields is accomplished by the Moyal product: $\{\phi\star\phi, (\partial^2\phi)\star\phi, \dots\}$. Then, a very natural starting point is to look at the renormalization of $(\phi\star\phi)(x)$. The Fourier transform in momentum space of the bare operator is

$$\widetilde{(\phi_0\star\phi_0)}(p) = \int \frac{d^Dx}{(2\pi)^D}\, e^{-ipx}(\phi_0\star\phi_0)(x) = \int \frac{d^Dq}{(2\pi)^D}\, e^{ip\wedge q}\widetilde{\phi}_0(q)\widetilde{\phi}_0(p-q)\,, \tag{2.2}$$

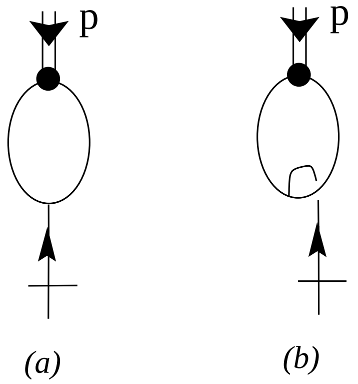


Figure 1: One-loop insertion in 1PI one-point function.

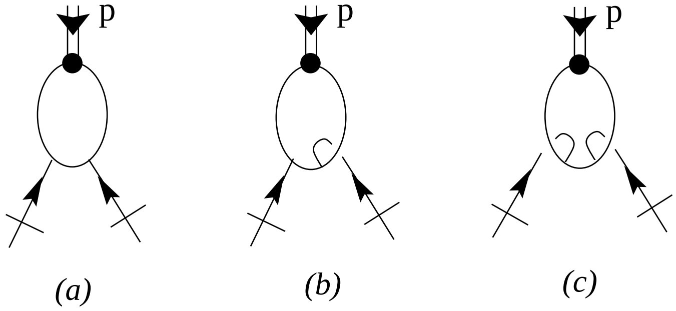
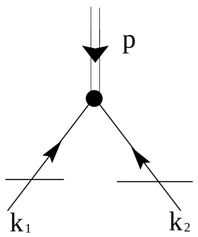


Figure 2: One-loop insertion in 1PI two-point function.

with $p \wedge q \equiv p^i \theta^{ij} q^j$. The insertion of this operator at tree level is



$$= (2\pi)^D \delta^D(p + k_1 + k_2)\ 2\cos(k_1 \wedge k_2)\,, \tag{2.3}$$

where the horizontal lines in the k_1 and k_2 momentum legs mean that their propagators have been amputated.

For a general diagram with superficial degree of divergence ω, the insertion of $\phi \star \phi$ decreases it to $\omega - 2$. In the six-dimensional ϕ^3 field theory, the superficial degree of divergence of an n-point function is $w = 2(n-3)$. Therefore, to analyze the renormalization of $\phi \star \phi$, we just have to consider its insertion at $n = 1, 2$. Since we only look at the connected Green functions with (amputated) external legs, we ignore the $n = 0$ case.[1]

The type of one-loop insertions in the 1PI one- and two-point functions are shown in figures 1 and 2. The UV divergences[2] of the planar graphs can be subtracted by

[1] If not, the identity operator enters into the renormalization.

[2] We use the Schwinger cut-off regularization [7].

the introduction of the renormalized composite operator

$$[\phi\star\phi] = \frac{g}{2^5\pi^3}\Lambda^2\phi_0 - \frac{g}{2^6\pi^3}\ln\left(\frac{\Lambda}{\mu}\right)\left(\frac{\partial^2}{6}+m^2\right)\phi_0 + \\ + \left(1-\frac{g^2}{2^8\pi^3}\ln\left(\frac{\Lambda}{\mu}\right)\right)\phi_0\star\phi_0 + \mathcal{O}\big(g^3\big)\,. \tag{2.4}$$

The planar graph in figure 1a produces the operator mixing with ϕ_0. To cancel the UV divergence of the graph in figure 2a one has to introduce a counter-term for $\phi_0\star\phi_0$. Observe that the renormalization of $\phi\star\phi$ preserves the $\star$-product structure. We could look at the renormalization of $[\phi\star\phi\star\phi]$ and obtain the same structure: it mixes with ϕ_0, $\phi_0\star\phi_0$ and $\phi_0\star\phi_0\star\phi_0$, because of the UV planar graphs with one, two and three legs (respectively) attached to the loop. The general result is that the basis of composite operators with the $\star$-product structure is closed under renormalization. We think that this fact is crucially related to the (possible) renormalizability of the non-commutative field theory.

Non-planar graphs. But the planar graphs are not the full story. For the one-point function, the non-planar contribution of figure 1b is

$$\left\langle \left[\widetilde{\phi\star\phi}\right](p)\widetilde{\phi}(k)\right\rangle_{NP} = -\frac{g}{2}(2\pi)^6\delta^6(p+k)\int\frac{d^6p}{(2\pi)^6}\frac{e^{ip\wedge k}}{(p^2+m^2)\left((p+k)^2+m^2\right)} \sim \\ \sim -\frac{g}{16\pi^3}(2\pi)^6\delta^6(p+k)\times \\ \times\left(\frac{2}{|\widetilde{k}|^2}+\left(\frac{k^2}{6}+m^2\right)\ln(|\widetilde{k}|\mu)+\cdots\right) \quad \text{when } |\widetilde{k}|\simeq 0\,. \tag{2.5}$$

The exact result for the integral can be obtained from $I_2(k;\widetilde{k})$ in appendix A. If the insertion of the composite operator is at zero momentum in the non-commutative directions, it appears precisely at the IR singularity $\widetilde{k}^i \equiv \theta^{ij}k^j = 0$.

The non-planar contribution to the amputated and connected two-point function is given by the diagrams in figures 1b, 2b and c,[3]

$$\left\langle \left[\widetilde{\phi\star\phi}\right](p)\widetilde{\phi}(k_1)\widetilde{\phi}(k_2)\right\rangle_{NP} = \frac{g^2}{2}(2\pi)^6\delta^6(p+k_1+k_2)\times \\ \times\left\{\cos(k_1\wedge k_2)\left(\frac{1}{p^2+m^2}I_2[p;\tilde{p}]+J_0[p,k_1;\tilde{p}]\right)+\right. \\ \left. + e^{ik_1\wedge k_2}J_0\left[p,k_1;2\widetilde{k}_1\right] + e^{-ik_1\wedge k_2}J_0\left[p,k_2;2\widetilde{k}_2\right]\right\}. \tag{2.6}$$

Again, for $\tilde{p}=0$, its insertion is singular. This situation generalizes to any operator insertion at zero momentum in the non-commutative directions. The singularities

[3] The functions $I_2[p;\widetilde{k}]$ and $J_0[p_1,p_2;\widetilde{k}]$ are defined in appendix A.

appear because of the high-momentum circulating in the non-planar loop. Therefore, also for composite operators there is the UV/IR mixing noticed in [7]. In this case, it is caused by the UV divergences associated with the composite operators.

Another way to understand this singularity is to observe that, contrary to the case of local QFT, a composite local operator $\mathcal{O}_i(x)$ and its corresponding coupling parameter g_i have *different* multiplicative renormalizations, with the difference parametrized precisely by the IR divergence of $\widetilde{\mathcal{O}}_i(p)$ at $\widetilde{p}=0$. If one restores the cutoff and then takes $\widetilde{p}=0$, one recovers the usual property

$$\int dx_{nc}\; g_{i,0}(\Lambda)\mathcal{O}_{i,\Lambda}(x) = \int dx_{nc}\; g_i(\mu)[\mathcal{O}_i]_\mu(x) + \text{lower dim. ops.} \tag{2.7}$$

So, it is crucial that the parameter g_i *only* couples to the spacetime integral of $\mathcal{O}_i$.

Besides this intrinsic IR singularity for the zero momentum insertion of $\widetilde{\mathcal{O}}$, a general n-point function will have a multiple set of singularities located at different linear combinations of the external momenta, $\sum_i a_i\widetilde{k}_i = 0$, with the constants a_i related to the different momentum channels or graphs. For instance, in (2.6) there are additional one-loop singularities located at $\widetilde{k}_1=0$ and $\widetilde{k}_2=0$.

Composite operators with some $\star$-products missing. Even though they are odd objects in NCQFT, one could ask about the quantum properties of composite operators which lack some $\star$-products, like $[\phi^2]$, $[(\phi\star\phi)\phi]$, etc. Formally, they could be expressed as an infinite sum of operators with every product given by the $\star$-product, which would correspond to working in the $\star$-product operator basis. But at the perturbative level we can look at, for instance, $[\phi^2](x)$ as a symbol, with its tree-level insertion just given by (2.3) without the phase factor. In this case, the one-loop integrals corresponding to figures 1 and 2 are all finite, since they always have a Moyal phase, coming from the interaction vertices, which cuts off the high frequency modes in the loop.[4] Then, at one-loop, $[\phi^2]$ only renormalizes with the identity operator. We can also look at $[\phi^3]$, defined by its obvious insertion in the three point function at zero order in the coupling constant. In this case, it is only UV divergent when inserted in the one-point function. It can be made finite simply by adding a counter-term proportional to ϕ_0. Finally, for the insertion of $[(\phi\star\phi)\phi]$, the necessary counter-terms to make it one-loop finite are ϕ_0, ϕ_0^2 and $(\phi_0\star\phi_0)\phi_0$.

Since the UV properties of these operators are different from the analogous operators with all the products given by the $\star$-product, we expect that the location of their singularities associated to the $\theta\to 0$ limit to be also different. Indeed, in general their insertion at zero momentum is still divergent.[5] But when inserted in a general n-point function, their additional singularities are located at different places from the ones corresponding to the insertions of the $\star$-product's operators. For instance, the insertion of $[\widetilde{\phi^2}]$ in the two-point function produces singularities at $\widetilde{k}_1 \pm \widetilde{k}_2 = 0$.

[4] We have verified this situation holds up to two-loops.

[5] Which in the case of $[\phi^2]$ is fully related to the UV divergence of the mass parameter.

Non-commutative $\lambda\varphi^4$ in four dimensions. Same qualitative results are obtained for the case of a four-dimensional $\lambda\varphi^4$ interaction,

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)\star(\partial\varphi) + \frac{1}{2}m^2\varphi\star\varphi + \frac{\lambda}{4!}\varphi\star\varphi\star\varphi\star\varphi\,. \tag{2.8}$$

In this case, the discrete symmetry $\varphi \to -\varphi$ prevents $\varphi\star\varphi$ from mixing with φ. Looking at its insertion into the (amputated and connected) two-point function at one-loop, we get the renormalization

$$[\varphi\star\varphi] = \left(1 - \frac{\lambda}{2^4 3\pi^2}\ln\left(\frac{\Lambda}{\mu}\right)\right)\varphi_0\star\varphi_0\,. \tag{2.9}$$

It does not correspond to the inverse multiplicative renormalization of the mass squared, which is

$$m^2 = \left(1 + \frac{\lambda}{2^3 3\pi^2}\ln\left(\frac{\Lambda}{\mu}\right)\right)m_0^2 - \frac{\lambda}{12\pi^2}\Lambda^2\,. \tag{2.10}$$

As explained before, the reason is a singularity for the insertion of $[\widetilde{\varphi\star\varphi}](p)$ at zero momentum in the non-commutative directions. For $\widetilde{p}\simeq 0$,

$$\begin{aligned}\left\langle\left[\frac{1}{2}\widetilde{\varphi\star\varphi}\right](p)\widetilde{\varphi}(k_1)\widetilde{\varphi}(k_2)\right\rangle_{NP} &\sim (2\pi)^4\delta^4(p+k_1+k_2)\cos(k_1\wedge k_2)\times\\ &\times\left(-\frac{\lambda}{2^4 3\pi^2}\right)\left(\frac{2}{|\widetilde{p}|}+\ln(|\widetilde{p}|\mu)\right),\end{aligned} \tag{2.11}$$

whose coefficient of the log divergence added to the coefficient of the UV log divergence in (2.9) exactly matches with the negative of the log coefficient in (2.10).

We also point out that, as in the six-dimensional ϕ^3 field theory, the additional one-loop singularities of $[\varphi\star\varphi]$ are located at $\widetilde{k}_1, \widetilde{k}_2 = 0$, and of $[\varphi^2]$ at $\widetilde{k}_1 - \widetilde{k}_2 = 0$. The one-loop insertion of the bare operator φ_0^2 in the connected two-point function is also finite.

3. Operator product expansion in NCQFT

OPE in local QFT. In NCQFT, the non-commutativity scale $\theta \neq 0$ changes the UV properties of the theory. As a consequence, renormalized quantities do not have a smooth $\theta \to 0$ limit; or in other words, the two limits $\Lambda \to \infty$ and $\theta \to 0$ do not commute [7].

There is a similar situation in local QFT: the product of two renormalized operators located at different spacetime points, $[\mathcal{O}_1](x)[\mathcal{O}_2](y)$ is singular for $x \to y$ if the

UV divergences of $[\mathcal{O}_1\mathcal{O}_2](x)$ are different from the divergences of $[\mathcal{O}_1](x)[\mathcal{O}_2](y)$. In this case, the physical meaning for the non-commutativity of the $x \to y$ and $\Lambda \to \infty$ limits is encoded in the Operator Product Expansion formula:

$$[\mathcal{O}_1]_\mu(x)[\mathcal{O}_2]_\mu(y) = \sum_{n=0}^{\infty} |y-x|^{d_n-d_1-d_2} C_{12}{}^n\left(|y-x|\mu; g(\mu)\right)[\mathcal{O}_n]_\mu(x)\,, \tag{3.1}$$

where $\{\mathcal{O}_n\}$ is a convenient basis of local composite operators, with canonical mass dimension d_n, and μ is the renormalization scale. The Wilson coefficients $C_{12}{}^n(|y-x|)$ can be computed perturbatively, finding that in general they are logarithmically divergent when $|y-x| \to 0$ (for field theories defined at a Gaussian fixed point). For instance, in the case of commutative $g\phi^3$ scalar theory in six dimensions, the product of two fundamental fields is

$$\begin{aligned}\phi\left(x-\frac{\epsilon}{2}\right)\phi\left(x+\frac{\epsilon}{2}\right) = {}& \frac{1}{|\epsilon|^2}C_\phi(|\epsilon|\mu; g)\phi(x) + C_{\partial^2\phi}(|\epsilon|\mu; g)\partial^2\phi(x) + \\ & + C_{m^2\phi}(|\epsilon|\mu; g)m^2\phi(x) + C_{[\phi^2]}(|\epsilon|\mu; g)[\phi^2](x) + \\ & + \epsilon^\mu\epsilon^\nu C_{[\partial^2\phi^2]}(|\epsilon|\mu; g)[\phi\partial_\mu\partial_\nu\phi - \partial_\mu\phi\partial_\nu\phi](x) + \mathcal{O}(\epsilon^4)\,,\end{aligned} \tag{3.2}$$

where a perturbative calculation gives the Wilson coefficients

$$\begin{aligned} C_\phi(|\epsilon|\mu; g) &= -\frac{g}{16\pi^3} + \mathcal{O}(g^3)\,, \\ C_{\partial^2\phi}(|\epsilon|\mu; g) &= \frac{g}{2^7 3\pi^3}\ln(|\epsilon|\mu) + \mathcal{O}(g^3)\,, \\ C_{m^2\phi}(|\epsilon|\mu; g) &= \frac{g}{2^6\pi^3}\ln(|\epsilon|\mu) + \mathcal{O}(g^3)\,, \\ C_{[\phi^2]}(|\epsilon|\mu; g) &= 1 - \frac{g^2}{2^6\pi^3}\ln(|\epsilon|\mu) + \mathcal{O}(g^4)\,. \end{aligned} \tag{3.3}$$

We want to revisit the same process in NCQFT: to analyze the possible singularities associated with the product of two fields and see if there is still an Operator Product Expansion parametrizing it.

Singularities in position space. As stated in the introduction, we will limit our analysis to one-loop and mainly use the six-dimensional ϕ^3 field theory as illustrative example. Fortunately, the results are already non trivial enough to derive some conclusions. We will consider an $2+n$-point Green function as a function of the distance $\epsilon = x - y$ between the position of the fields $\phi(x)$ and $\phi(y)$. The rest of the fields in the Green function will be Fourier transformed to momentum space, with their external propagators amputated.

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We can start with the connected three-point function. Its lowest order contribution can be easily computed

$$
\begin{aligned}
&\left\langle \phi\left(x-\frac{\epsilon}{2}\right)\phi\left(x+\frac{\epsilon}{2}\right)\widetilde{\phi}(k)\right\rangle = \\
&= -\frac{g}{2}e^{-ik(x-\epsilon/2)}\int \frac{d^6p}{(2\pi)^6}\frac{e^{ip\epsilon}\left(e^{ip\wedge k}+e^{-ip\wedge k}\right)}{(p^2+m^2)\left((p+k)^2+m^2\right)} \\
&= -ge^{-ik(x-\epsilon/2)}\int_0^1 \frac{d\alpha}{16\pi^2}\left(\frac{\Delta}{2\pi\left|\widetilde{k}+\epsilon\right|}K_1\left[\left|\widetilde{k}+\epsilon\right|\Delta\right]+\frac{\Delta}{2\pi\left|\widetilde{k}-\epsilon\right|}K_1\left[\left|\widetilde{k}-\epsilon\right|\Delta\right]\right),
\end{aligned}
\tag{3.4}
$$

where $\Delta^2 = k^2\alpha(1-\alpha)+m^2$, $\widetilde{k}^i = \theta^{ij}k^j$ and $K_n[z]$ is the second kind Bessel function of order n. Most of the results in this section already appear in this simple example. First, notice that (3.4) is finite for $\widetilde{k}\pm\epsilon\neq 0$. When $\widetilde{k}\pm\epsilon\simeq 0$, we have

$$
\begin{aligned}
&\left\langle \phi\left(x-\frac{\epsilon}{2}\right)\phi\left(x+\frac{\epsilon}{2}\right)\widetilde{\phi}(k)\right\rangle \sim \\
&\qquad \sim -\frac{g}{64\pi^3}e^{-ik(x-(\epsilon/2))}\left(\frac{2}{|\widetilde{k}\pm\epsilon|^2}+\left(\frac{k^2}{6}+m^2\right)\ln\left(\left|\widetilde{k}\pm\epsilon\right|\mu\right)\right).
\end{aligned}
\tag{3.5}
$$

Again, the dimensionfull scale $\theta\neq 0$ mixes UV and IR effects. Sending first $\epsilon\to 0$ and then $\widetilde{k}\to 0$, the divergence in (3.5) is interpreted as IR. If we reverse the order of the limits the same divergence has a UV (short distance effect) interpretation.

In fact, *the length scale ϵ is replaced by the combination $\epsilon\pm\widetilde{k}$.* The product of two fields ϕ, instead of being singular when they are evaluated at the same spacetime point (the situation of local QFT), is now singular when the distance between them is proportional to the momentum in the non-commutative directions of the additional external field. This result supports the picture of having extended objects whose characteristic size is proportional to its momentum [30]–[33].

We repeat the same calculations for the case of the four-point function at one-loop. We get

$$
\begin{aligned}
&\left\langle \phi\left(x-\frac{\epsilon}{2}\right)\phi\left(x+\frac{\epsilon}{2}\right)\widetilde{\phi}(k_1)\widetilde{\phi}(k_2)\right\rangle = \\
&= -\frac{g^2}{4}e^{-iK_+(x-(\epsilon/2))}\left\{\frac{2\cos(k_1\wedge k_2)}{K_+^2+m^2}\left(I_2\left[K_+;\widetilde{K}_++\epsilon\right]+I_2\left[K_+;\widetilde{K}_+-\epsilon\right]\right)+\right. \\
&\qquad + e^{ik_1\wedge k_2}\left(J_0\left[K_+,k_1;\widetilde{K}_++\epsilon\right]+J_0\left[K_+,k_2;\widetilde{K}_+-\epsilon\right]\right)+ \\
&\qquad + e^{-ik_1\wedge k_2}\left(J_0\left[K_+,k_2;\widetilde{K}_++\epsilon\right]+J_0\left[K_+,k_1;\widetilde{K}_+-\epsilon\right]\right)+ \\
&\qquad + e^{ik_1\wedge k_2}\left(J_0\left[K_+,k_1;\widetilde{K}_-+\epsilon\right]+J_0\left[K_+,k_2;\widetilde{K}_--\epsilon\right]\right)+ \\
&\qquad \left. + e^{-ik_1\wedge k_2}\left(J_0\left[K_+,k_2;\widetilde{K}_-+\epsilon\right]+J_0\left[K_+,k_1;\widetilde{K}_--\epsilon\right]\right)\right\},
\end{aligned}
\tag{3.6}
$$

where $K_\pm = k_1\pm k_2$. As before, the expression is finite, unless $\widetilde{K}_\pm\pm\epsilon = 0$.

In local QFT, the singularity associated with the product of two local operators always appears at the invariant point $\epsilon = 0$. It allows one to define an OPE formula, where the $\epsilon = 0$ singularity can be encoded in the universal Wilson coefficients. On the contrary, we just saw that in NCQFT the length scale ϵ is mixed, via the non-commutativity scale, with the momenta flowing into the Green function. In general, for each graph of a given $2 + n$-point correlation function with fixed momenta k_i for the n-external legs, the previous local singularity equation $\epsilon = 0$ is replaced by $\epsilon + \sum a_i \widetilde{k}_i = 0$, with coefficients a_i depending on the particular graph. This momentum-dependent shift has dramatic consequences for the old OPE formula. Now, the effects of the short distance scale ϵ cannot be decoupled and codified in some Wilson coefficients in front of the insertion of single local operators.

Notice that the Taylor expansion in ϵ for the finite expression on the right hand side of (3.4) and (3.6) is obviously reproduced by the corresponding ϵ expansion of $\phi(x-\epsilon/2)\phi(x+\epsilon/2)$ on the left hand side. But this is far from proving the existence of an OPE formula. In particular, the UV/IR singularities of the composite operators and an inconsistent RG μ dependence, already at one-loop, for the renormalized fields ϕ and $[\phi^2]$, invalidates the naive ϵ expansion for $\phi(x-\epsilon/2)\phi(x+\epsilon/2)$. Wilson's OPE has more to do with the relation between the analytic properties of the Green functions and the decoupling of the UV scales. In fact, the failure of the OPE in NCQFT is easily seen in momentum space, due to the simple expression of the $\star$-product in that representation.

OPE in momentum space. One can Fourier transform (3.1) to momentum space[6]

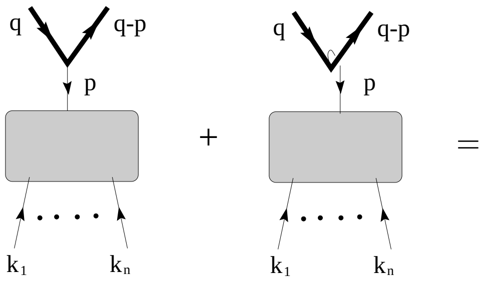
$$\widetilde{\mathcal{O}}_1(q)\widetilde{\mathcal{O}}_2(p-q) = \sum_n \widetilde{C}_{12}{}^n(q)\widetilde{\mathcal{O}}_n(p)\,. \tag{3.7}$$

The decoupling of the scales q and p in this formula is intimately related to the Wilson renormalization group approach to field theory. The Fourier transformed Wilson coefficients scale as $|q|^{d_n-d_1-d_2}$, making the OPE extremely useful in the regime of $|q| \to \infty$, where only the lowest-dimensional operators are the most relevant ones. As we know, for generic q the expansion (3.7) can be full of renormalon singularities which spoil its Borel summability. But the $|q| \to \infty$ limit makes it legitimate to use the OPE.

How much of this still holds in NCQFT? Consider a general connected $n+2$-point function in momentum representation. Take the momentum of one leg to be q and another one $p-q$ and consider the regime where $|q|$ is much larger than any mass scale in the Green function. One possibility is that the two-external legs meet at the same interaction vertex. In this case we have (the high momentum q flows

[6]Even though in this process the constant terms in $C_{12}{}^n(|x-y|)$ are lost unless $q = 0$.

through the thicker lines)

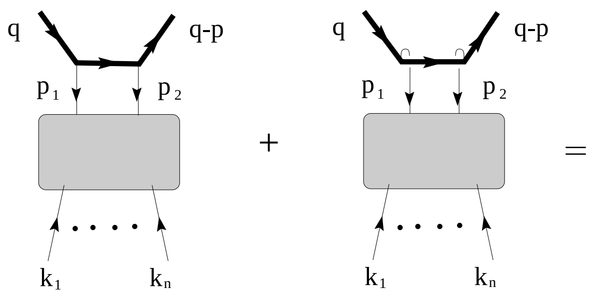


$$= \cos(q \wedge p) \left(\frac{1}{q^2 + m^2} \right) \left(\frac{1}{(q-p)^2 + m^2} \right) G_{n+1}(p, k_1, ..., k_2)$$

$$= \cos(q \wedge p) \left(\frac{1}{q^4} - \frac{2q \cdot p}{q^6} + \cdots \right) G_{n+1}(p, k_1, ..., k_2)\,. \tag{3.8}$$

As usual, the propagators carrying the high momenta q can be expanded in powers of $|p \cdot q||q|^{-2}, m^2|q|^{-2} \ll 1$, which can be re-interpreted as the insertion of derivatives and mass multiplications of the single field $\widetilde{\phi}(p)$. In this case, a weaker version of the OPE formula would hold, with the Wilson coefficients being simply multiplied by $\cos(q \wedge p)$. The question is whether or not there is a single operator insertion with the net momentum p.

This possibility is eliminated when the high momentum between the two-external legs flows through an intermediate leg. In this case, we have different ways to share the total momentum insertion $p = p_1 + p_2$. Without having to analyze the case where there is a loop momentum flowing between p_1 and p_2,[7] we can easily identify problems with a universal OPE in terms of single local operators with momentum p. If we look at the diagrams of the sort



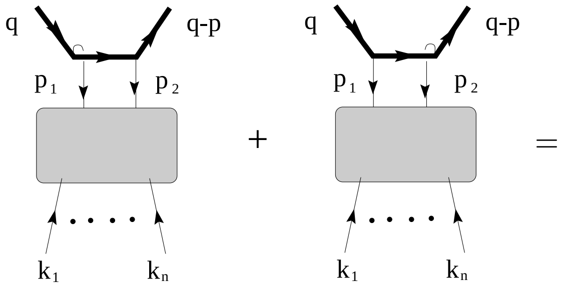
$$= \cos(q \wedge p) \cos(p_1 \wedge p_2) \times \tag{3.9}$$

$$\times \left(\frac{1}{q^2 + m^2} \right) \left(\frac{1}{(q-p)^2 + m^2} \right) \left(\frac{1}{(q+p_1)^2 + m^2} \right) G_{n+2}(p_1, p_2, k_1, ..., k_2)\,,$$

we still have the same global factor $\cos(q \wedge p)$. The factor $\cos(p_1 \wedge p_2)$ could be explained as coming from the insertion of $\phi \star \phi$ and convenient derivatives of it. But

[7] Which certainly would be a necessary step in order to proof an OPE formula.

there are more graphs, the ones given by crossing only one-low-momentum leg with the high-momentum ones:



$$
= \cos(q \wedge (p_2 - p_1) + p_1 \wedge p_2) \left(\frac{1}{q^2 + m^2} \right) \times
$$
$$
\times \left(\frac{1}{(q-p)^2 + m^2} \right) \left(\frac{1}{(q-p_1)^2 + m^2} \right) G_{n+2}(p_1, p_2, k_1, ..., k_2) \, . \quad (3.10)
$$

From these we see that, due to the presence of the arbitrary momentum $p_2 - p_1$ into the overall phase in (3.10), there is no way to reproduce it as a series of insertions of single composite operators $\widetilde{\mathcal{O}}_n(p)$.

4. Conclusions

Essentially, there are three results reported in this paper.

In section 2, we concluded that if the renormalization program (à la Dyson) for n-point correlation functions can be performed at higher loops, then the same would be valid for single insertions of composite operators where all the products are given by $\star$-products. We obtained that, as in commutative QFT, the composite operators require their own renormalization, with the possibility of getting mixed. Then, we have found their insertion at zero momentum in the non-commutative directions to be generically singular. The reason being that, contrary to the commutative QFT, their renormalization (at $\widetilde{p} \neq 0$) does not correspond to the renormalization of the associated coupling parameters.

By the one-loop analysis in section 3, we saw that the singularity associated with the product of two fields in scalar NCQFT is shifted by an amount proportional to the momentum flowing into the graphs. This provides another manifestation of the UV/IR mixing in non-commutative field theories.

Finally, an explicit check in momentum space for the non-commutative ϕ^3 field theory showed a breakdown of the OPE as a replacement of the operator product by a series insertion of single local operators.

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A. Useful one-loop integrals

$$
\begin{aligned}
I_{D-4}\left(q;\widetilde{k}\right) &= \int \frac{d^Dp}{(2\pi)^D}\,\frac{e^{ip\widetilde{k}}}{(p^2+m^2)\left((p+q)^2+m^2\right)} = \\
&= \int_0^1 \frac{d\alpha}{8\pi^2}\, e^{-i\alpha q\widetilde{k}} \left(\frac{\sqrt{q^2\alpha(1-\alpha)+m^2}}{2\pi|\widetilde{k}|}\right)^{D/2-2} \times \\
&\qquad \times K_{(D/2)-2}\left[\left|\widetilde{k}\right|\sqrt{q^2\alpha(1-\alpha)+m^2}\right]. \qquad \text{(A.1)}
\end{aligned}
$$

$$
\begin{aligned}
J_{D-6}\left(q_1,q_2;\widetilde{k}\right) &= \int \frac{d^Dp}{(2\pi)^D}\,\frac{e^{ip\widetilde{k}}}{(p^2+m^2)\left((p+q_1)^2+m^2\right)\left((p+q_2)^2+m^2\right)} = \\
&= \frac{1}{32\pi^3}\int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2\; e^{-i(q_1\alpha_1+q_2\alpha_2)\widetilde{k}}\Delta^{D-6} \times \\
&\qquad \times \left(2\pi\left|\widetilde{k}\right|\Delta\right)^{3-\frac{D}{2}} K_{\frac{D}{2}-3}\left[\left|\widetilde{k}\right|\Delta\right], \qquad \text{(A.2)}
\end{aligned}
$$

where $\Delta^2 = q_1^2\alpha_1(1-\alpha_1) + q_2^2\alpha_2(1-\alpha_2) + 2q_1q_2\alpha_1\alpha_2 + m^2$.

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